SKEW-PRODUCTS OF HIGHER-RANK GRAPHS AND CROSSED PRODUCTS BY SEMIGROUPS

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ABSTRACT. We consider a free action of an Ore semigroup on a higher-rank graph, and the induced action by endomorphisms of the C^*-algebra of the graph. We show that the crossed product by this action is stably isomorphic to the C^*-algebra of a quotient graph. Our main tool is Laca's dilation theory for endomorphic actions of Ore semigroups on C^*-algebras, which embeds such an action in an automorphic action of the enveloping group on a larger C^*-algebra.

1. Introduction

Kumjian and Pask [9] proved that if a group G acts freely on a directed graph E, then the associated crossed product $C^*(E) \rtimes G$ of the graph algebra is stably isomorphic to the graph algebra $C^*(G\backslash E)$ of the quotient graph. Their theorem has been extended in several directions: to actions of groups on higher-rank graphs ([10, Theorem 5.7] and [14, Corollary 7.5]), and to actions of Ore semigroups on directed graphs [13]. Here we consider actions of Ore semigroups on higher-rank graphs.

Our main theorem directly extends that of [13] to higher-rank graphs, but our proof has some interesting new features. First of these is our more efficient use of Laca's dilation theory for endomorphic actions [12]: by exploiting his uniqueness theorem, we have been able to bypass the complicated direct-limit constructions used in [13]. Second, we have found an explicit isomorphism. In searching for explicit formulas, we have revisited the case of group actions, and we think a third feature of general interest is our direct approach to crossed products of the C^*-algebras of skew-product graphs, which is based on the treatment of skew products of directed graphs in [6, §3] (see Theorem 5.1).

After a brief discussion of notation and background material, we discuss higher-rank graphs and their C^*-algebras in §2, and prove two general lemmas about the C^*-algebras of higher-rank graphs. In §3, we prove our first results about actions of semigroups, including a version of the Gross-Tucker theorem which will allow us to replace the underlying graph with a skew product. In §4 we apply Laca's dilation theory to higher-rank graph algebras. In §5 we discuss group actions on skew products, and then in §6 we pull the pieces together and prove our main theorem.

Background and notation. All semigroups in this paper are countable and have an identity 1. An *Ore semigroup* is a cancellative semigroup such that for all pairs $t, u \in S$, there exist $x, y \in S$ such that $xt = yu$. Ore and Dubreil proved that a semigroup is

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Ore if and only if it can be embedded in a group Γ such that $\Gamma = S^{-1}S$; the group Γ is unique up to isomorphism, and we call it the *enveloping group* of S.

An action of a semigroup S on a C^*-algebra A is an identity-preserving homomorphism α of S into the semigroup $\operatorname{End} A$ of endomorphisms of A. A *covariant representation* of (A, S, α) in in a C^*-algebra B consists of a nondegenerate homomorphism $\pi \to B$ and a homomorphism V of S into the semigroup of isometries in $M(B)$, such that $\pi(\alpha_t(a)) = V_t\pi(a)V_t^*$ for $a \in A$ and $t \in S$. The *crossed product* $A \times_\alpha S$ is generated by a universal covariant representation (i_A, i_S) in $A \times_\alpha S$. (In the recent literature, this is called the "Stacey crossed product".) When $S = \Gamma$, the endomorphisms are automorphisms, and we recover the usual crossed product $(A \rtimes_\alpha \Gamma, i_A, i_\Gamma)$. If (π, V) is a covariant representation of (A, S, α) in B, then we write $\pi \times V$ for the homomorphism of $A \times_\alpha S$ into B such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_S = V$.

To talk about stable isomorphisms, we need to consider tensor products with the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. Since $\mathcal{K}(\mathcal{H})$ is nuclear, there is no ambiguity in writing $A \otimes \mathcal{K}(\mathcal{H})$. However, we are interested in C^*-algebras which have universal properties, and we view $A \otimes \mathcal{K}(\mathcal{H})$ as the maximal tensor product $A \otimes_{\max} \mathcal{K}(\mathcal{H})$ which is universal for pairs of commuting representations of A and $\mathcal{K}(\mathcal{H})$ (see [17, Theorem B.27]).

We write λ and ρ for the left- and right-regular representations of a group Γ on $l^2(\Gamma)$, and $\{e_g : g \in \Gamma\}$ for the usual orthonormal basis of point masses. For $F \subset \Gamma$, χ_F is the operator on $l^2(\Gamma)$ which multiplies by the characteristic function of F, and $\chi_g := \chi_{\{g\}}$. We often use the relations $\lambda_h\chi_g = \chi_{hg}\lambda_h$ and $\rho_k\chi_g = \chi_{gk^{-1}}\rho_k$. When S is a subsemigroup of Γ, we identify $l^2(S)$ with the subspace $\overline{\operatorname{span}}\{e_t : t \in S\}$ of $l^2(\Gamma)$, and then $t \mapsto \lambda_t^S := \lambda_t|_{l^2(S)}$ is the usual Toeplitz representation of S on $l^2(S)$.

2. Higher-rank graphs and their C^*-algebras

Suppose $k \in \mathbb{N}$ and $k \geq 1$. A *graph of rank k*, or *k-graph*, is a countable category Λ with domain and codomain maps r and s, together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^k$, there is a unique pair (μ, ν) in $\Lambda \times \Lambda$ such that $s(\mu) = r(\nu)$, $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We write Λ^0 for the set of objects, and observe that the factorisation property allows us to identify Λ^0 with $d^{-1}(0)$; then we write $\Lambda^n := d^{-1}(n)$ for $n \in \mathbb{N}^k$. Visualisations of k-graphs are discussed in [16] and [15, Chapter 10]: we think of Λ^0 as the set of vertices, and $\lambda \in \Lambda^n$ as a path of degree n from $s(\lambda)$ to $r(\lambda)$.

As in [10], we assume throughout that our k-graphs are *row-finite* and *have no sources*, in the sense that $v\Lambda^n := r^{-1}(v) \cap \Lambda^n$ is finite and nonempty for every $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

Given a k-graph Λ, a *Cuntz-Krieger Λ-family* in a C^*-algebra B consists of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ in B satisfying

(CK1) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(CK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$;
(CK4) $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The graph algebra $C^*(\Lambda)$ is generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}$. When there is more than one graph around, we sometimes write $\{s_\lambda^\Lambda\}$ for emphasis. Each vertex

projection s_v (and hence by (CK3) each s_λ) is non-zero [10, Proposition 2.11], and

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\} \quad \text{(see [10, Lemma 3.1])}.$$

Lemma 2.1. *Suppose that Λ is a row-finite k-graph with no sources, that $m \in \mathbb{N}^k$, and that V is a subset of Λ^m such that the paths in V all have different sources. Let $\{F_n\}$ be an increasing sequence of finite subsets of V such that $V = \bigcup_n F_n$. Then each $s_n := \sum_{\mu \in F_n} s_\mu$ is a partial isometry, and there is a partial isometry $s_V \in M(C^*(\Lambda))$ such that $s_n \to s_V$ strictly. The limit s_V is independent of the choice of F_n, and satisfies*

$$s_V s_\alpha s_\beta^* = \begin{cases} s_{\mu\alpha} s_\beta^* & \text{if } r(\alpha) = s(\mu) \text{ for some } \mu \in V \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and, for paths β with $d(\beta) \geq m$,

$$s_\alpha s_\beta^* s_V = \begin{cases} s_\alpha s_{\beta'}^* & \text{if } \beta = \mu\beta' \text{ for some } \mu \in V \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

If $V \subset \Lambda^m$ and $W \subset \Lambda^p$ are two such sets, then $s_V s_W$ is the partial isometry s_{VW} associated to the set $VW := \{\mu\nu : \mu \in V,\ \nu \in W \text{ and } s(\mu) = r(\nu)\}$.

Proof. Since all the μ have the same degree, (CK3) and (CK4) imply that

$$s_n^* s_n = \sum_{\mu,\nu \in F_n} s_\mu^* s_\nu = \sum_{\mu \in F_n} s_\mu^* s_\mu = \sum_{\mu \in F_n} s_{s(\mu)};$$

since $s(\mu) \neq s(\nu)$ for $\mu \neq \nu$ in V, this is a sum of mutually orthogonal projections, and hence is a projection. Thus s_n is a partial isometry. For $\alpha, \beta \in \Lambda$, we have

$$s_n s_\alpha s_\beta^* = \begin{cases} s_{\mu\alpha} s_\beta^* & \text{if } r(\alpha) = s(\mu) \text{ for some } \mu \in F_n \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

If $r(\alpha) = s(\mu)$ for some $\mu \in V$, then $\mu \in F_n$ for large n, and hence the right-hand side of (2.3) is eventually constant for every $s_\alpha s_\beta^*$. Now an $\epsilon/3$ argument implies that $\{s_n a\}$ is Cauchy for every $a \in C^*(\Lambda)$. A similar calculation shows that $s_\alpha s_\beta^* s_n$ is eventually constant whenever $d(\beta) \geq m$. However, (CK4) and (CK2) imply that

$$\operatorname{span}\left\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda\right\} = \operatorname{span}\left\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda,\ d(\beta) \geq m\right\}$$

so $s_\alpha s_\beta^* s_n$ is eventually constant for all α, β, and we deduce as before that $\{a s_n\}$ is Cauchy for all $a \in C^*(\Lambda)$. Since $M(C^*(\Lambda))$ is complete in the strict topology [2, Proposition 3.6], we deduce that s_n converges strictly to a multiplier s_V. Then (2.3) implies (2.1), and similarly for (2.2).

The formula (2.1) implies that s_V is independent of the choice of sequence $\{F_n\}$. For $\alpha, \beta \in \Lambda$, (2.1) and the adjoint of (2.2) show that $s_V s_V^* s_V s_\alpha^* s_\beta = 0 = s_V s_\alpha s_\beta^*$ unless $r(\alpha) = s(\mu)$ for some $\mu \in V$, and in that case

$$s_V s_V^* s_V s_\alpha^* s_\beta = s_V s_V^* s_{\mu\alpha} s_\beta^* = s_V s_\alpha s_\beta^*;$$

either way, we have $s_V s_V^* s_V s_\alpha s_\beta^* = s_V s_\alpha s_\beta^*$. Thus $s_V s_V^* s_V = s_V$, and s_V is a partial isometry. The final assertion follows from two applications of (2.1). $\square$

Remark 2.2. Lemma 2.1 applies when $m = 0$, in which case the summands are projections and so is the limit s_V. To emphasise this, we write p_V for s_V when $m = 0$.

A k-graph morphism $\pi : \Lambda \to \Sigma$ is *saturated* if $r(\sigma) \in \pi(\Lambda^0) \Longrightarrow \sigma \in \pi(\Lambda)$. Recall (from [1], for example) that a homomorphism ϕ from a C^*-algebra to a multiplier algebra $M(B)$ is *extendible* if there are an approximate identity $\{e_i\}$ for A and a projection $p \in M(B)$ such that $\phi(e_i)$ converges strictly to p. If so, there is a unique extension $\overline{\phi} : M(A) \to M(B)$, which satisfies $\overline{\phi}(1) = p$ and is strictly continuous. Nondegenerate homomorphisms are extendible with $\overline{\phi}(1) = 1$.

Lemma 2.3. *Suppose that $\pi : \Lambda \to \Sigma$ is an injective saturated k-graph morphism between row-finite graphs with no sources. Then there is a homomorphism $\pi_* : C^*(\Lambda) \to C^*(\Sigma)$ such that $\pi_*(s^\Lambda_\lambda) = s^\Sigma_{\pi(\lambda)}$, and π_* is injective and extendible with $\overline{\pi_*}(1) = p_{\pi(\Lambda^0)}$. The assignment $\pi \mapsto \pi_*$ is functorial: $(\pi \circ \tau)_* = \pi_* \circ \tau_*$.*

Proof. Saturation means that $\{\sigma \in \Sigma : r(\sigma) = \pi(v)\} = \{\pi(\lambda) : r(\lambda) = v\}$ for every $v \in \Lambda^0$, so the Cuntz-Krieger relation (CK4) in Σ implies the analogous relation for the family $\{s^\Sigma_{\pi(\lambda)} : \lambda \in \Lambda\}$. Thus $\{s^\Sigma_{\pi(\lambda)}\}$ is a Cuntz-Krieger Λ-family, and there is a homomorphism π_* satisfying $\pi_*(s^\Lambda_\lambda) = s^\Sigma_{\pi(\lambda)}$. Since π is injective and every $s^\Sigma_w \neq 0$, the gauge-invariant uniqueness theorem [10, Theorem 3.4] implies that π_* is faithful.

To see that π_* is extendible, write $\Lambda^0 = \bigcup_n F_n$ as an increasing union of finite sets. Then $p_n := \sum_{v \in F_n} s^\Lambda_v$ is an approximate identity for $C^*(\Lambda)$. The images $\pi(F_n)$ satisfy $\bigcup_n \pi(F_n) = \pi(\Lambda^0)$, and since π is injective,

$$\pi(p_n) = \sum_{v \in F_n} p^\Sigma_{\pi(v)} = \sum_{w \in \pi(F_n)} p^\Sigma_w,$$

which by Lemma 2.1 converge strictly to $p_{\pi(\Lambda^0)}$. Thus π_* is extendible with $\overline{\pi_*}(1) = p_{\pi(\Lambda^0)}$. The functoriality follows from the formula $\pi_*(s^\Lambda_\lambda) = s^\Sigma_{\pi(\lambda)}$. □

3. A Gross-Tucker theorem

Suppose that α is a left action of an Ore semigroup S on a k-graph Σ, and that α is *free* in the sense that $\alpha_t(\lambda) = \alpha_u(\lambda)$ implies $t = u$. (It suffices to check freeness on vertices.) We will show that if α admits a fundamental domain, then there is an isomorphism of Σ onto a skew product which carries α into a canonical action of S by left translation. Such results were first proved for actions of groups on directed graphs by Gross and Tucker (see, for example, [4, Theorem 2.2.2]), and when S is a group, Theorem 3.2 below was proved by Kumjian and Pask [10, Remark 5.6].

Even the first step, which is the construction of the quotient graph, relies on the Ore property. We define a relation $\sim$ on Σ by

$$\lambda \sim \mu \Longleftrightarrow \text{ if there exist } t, u \in S \text{ such that } \alpha_t(\lambda) = \alpha_u(\mu).$$

The relation $\sim$ is trivially reflexive and symmetric. To see that it is transitive, suppose $\lambda \sim \mu$ and $\mu \sim \nu$, so that there exist $s, t, u, v \in S$ such that $\alpha_s(\lambda) = \alpha_t(\mu)$ and $\alpha_u(\mu) = \alpha_v(\nu)$. Since S is Ore, there exist $x, y \in S$ such that $xt = yu$. Then $\alpha_{xs}(\lambda) = \alpha_{xt}(\mu) = \alpha_{yu}(\mu) = \alpha_{yv}(\nu)$, which implies that $\lambda \sim \nu$. Thus $\sim$ is an equivalence relation on Σ. Since equivalent elements have the same degree, it makes sense to write $(S\backslash\Sigma)^0$ for the set of equivalence classes of vertices, $S\backslash\Sigma$ for the set of all equivalence classes, and

to define $d : S\backslash\Sigma \to \mathbb{N}^k$ by $d([\lambda]) = d(\lambda)$. It is easy to check that there are well-defined maps $r, s : S\backslash\Sigma \to (S\backslash\Sigma)^0$ such that $r([\lambda]) = [r(\lambda)]$ and $s([\lambda]) = [s(\lambda)]$.

Lemma 3.1. *With notation as above* $((S\backslash\Sigma)^0, S\backslash\Sigma, r, s, d)$ *is a k-graph, with composition defined by*

$$[\lambda][\mu] = [\alpha_t(\lambda)\alpha_u(\mu)] \text{ where } t, u \in S \text{ satisfy } \alpha_t(s(\lambda)) = \alpha_u(r(\mu)), \tag{3.1}$$

and $q : \lambda \mapsto [\lambda]$ *is a k-graph morphism.*

Proof. To verify that $S\backslash\Sigma$ is a k-graph, we have to check that:

- the right-hand side of (3.1) is independent of the choice of t and u (this uses the Ore property and the freeness of the action);
- the right-hand side of (3.1) is independent of the choice of coset representatives: (this uses the Ore property);
- $r([\lambda][\mu]) = r([\lambda])$ and $s([\lambda][\mu]) = s([\mu])$;
- associativity (this uses the Ore property);
- the classes $[\iota_v]$ have the properties required of the identity morphisms at $[v]$;
- $S\backslash\Sigma$ has the factorisation property.

Finally, if λ and μ are composable, we can take $t = u = 1$ in (3.1), and deduce that $q(\lambda\mu) = q(\lambda)q(\mu)$. □

Now suppose that Λ is a k-graph and $\eta : \Lambda \to S$ is a functor into a semigroup S (viewed as a category with one object). As in [10, Definition 5.1], we can make the set-theoretic product $\Lambda \times S$ into a k-graph $\Lambda \times_\eta S$ by taking $(\Lambda \times_\eta S)^0 = \Lambda^0 \times S$, defining $r, s : \Lambda \times_\eta S \to (\Lambda \times_\eta S)^0$ by

$$r(\lambda, t) = (r(\lambda), t) \quad \text{and} \quad s(\lambda, t) = (s(\lambda), t\eta(\lambda)),$$

defining the composition by

$$(\lambda, t)(\mu, u) = (\lambda\mu, t) \quad \text{when } s(\lambda, t) = r(\mu, u) \text{ (which is equivalent to } u = t\eta(\lambda)\text{)},$$

and defining $d : \Lambda \times_\eta S \to \mathbb{N}^k$ by $d(\lambda, t) = d(\lambda)$. Of course, one has to check the axioms to see that this does define a k-graph, but this is routine. We call $S \times_\eta S$ a *skew product.* Every $\Lambda \times_\eta S$ carries a natural action lt of S defined by $\mathrm{lt}_u(\lambda, t) = (\lambda, ut)$, and this action is free because S is cancellative.

The Gross-Tucker theorem implicit in [10, Remark 5.6] says that every free action of a group Γ on a k-graph Σ is isomorphic to the action lt on a skew-product $(\Gamma\backslash\Sigma) \times_\eta \Gamma$. As in [13], to get a Gross-Tucker theorem for actions of an Ore semigroup S, one has to insist that the action admits a *fundamental domain*, which is a subset F of Σ such that for every $\sigma \in \Sigma$ there are exactly one $\mu \in F$ and one $t \in S$ such that $\alpha_t(\mu) = \sigma$, and such that $r(\mu) \in F$ for every $\mu \in F$.

For a skew product $\Lambda \times_\eta S$, $F = \{(\lambda, 1_S) : \lambda \in \Lambda\}$ is a fundamental domain. The following "Gross-Tucker Theorem" says that existence of a fundamental domain characterises the actions lt on skew products.

Theorem 3.2. *Suppose that Σ is a row-finite k-graph with no sources, and α is a free action of an Ore semigroup S on Σ which admits a fundamental domain F. Let*

$q : \Sigma \to S\backslash\Sigma$ be the quotient map, and define $c : S\backslash\Sigma \to F$, $\eta : S\backslash\Sigma \to S$ and $\xi : \Sigma \to S$ by

$$q(c(\lambda)) = \lambda, \quad s(c(\lambda)) = \alpha_{\eta(\lambda)}(c(s(\lambda))) \quad \text{and} \quad \sigma = \alpha_{\xi(\sigma)}(c(q(\sigma))). \tag{3.2}$$

Then $\eta : S\backslash\Sigma \to S$ is a functor, and the map $\phi(\sigma) := (q(\sigma), \xi(\sigma))$ is an isomorphism of Σ onto the skew product $(S\backslash\Sigma) \times_\eta S$, with inverse given by $\phi^{-1}(\lambda, t) = \alpha_t(c(\lambda))$. The isomorphism ϕ satisfies $\phi \circ \alpha_t = \mathrm{lt}_t \circ\phi$.

When S is a group, every free action of S admits a fundamental domain, and we recover the result of [10, Remark 5.6]. Indeed, that proof starts by constructing a suitable fundamental domain. The rest of the proof of [10, Remark 5.6] then carries over to our situation, and shows that the formula we give for ϕ^{-1} defines an isomorphism of $(S\backslash\Sigma) \times_\eta S$ onto Σ.

Example 3.3. There are free semigroup actions which do not admit a fundamental domain. For example, consider the k-graph Δ_k of [11, §3], which has vertex set $\Delta_k^0 = \mathbb{Z}^k$, morphisms $\{(m, n) \in \mathbb{Z}^k \times \mathbb{Z}^k : m \le n\}$, $r(m, n) = m$, $s(m, n) = n$, composition given by $(m, n)(n, p) = (m, p)$, and degree map $d : (m, n) \mapsto n - m$. There is a free action α of $\mathbb{N}^k$ on Δ_k such that $\alpha_p(m, n) = (m + p, n + p)$, and we claim that this action cannot have a fundamental domain. To see this, note that a fundamental domain F would have to contain, for every $m \in \Delta_k^0$, a vertex $n \le m$ (so that $m = \alpha_{m-n}(n)$ for some $n \in F$). Thus it would have to contain infinitely many vertices. But if F has just two distinct vertices n, p, then every $m \ge n \vee p$ can be written as $m = \alpha_{m-n}(n) = \alpha_{m-p}(p)$. So there is no fundamental domain.

4. Dilating semigroup actions

Theorem 4.1 (Laca). *Suppose that S is an Ore semigroup with enveloping group $\Gamma = S^{-1}S$, and $\alpha : S \to \operatorname{End} A$ is an action of S by injective extendible endomorphisms of a C^*-algebra A.*

(a) *There are an action β of Γ on a C^*-algebra B and an injective extendible homomorphism $j : A \to B$ such that*

(L1) *$j \circ \alpha_u = \beta_u \circ j$ for $u \in S$, and*

(L2) *$\bigcup_{u\in S} \beta_u^{-1}(j(A))$ is dense in B;*

the triple (B, β, j) with these properties is unique up to isomorphism.

(b) *Suppose (B, β, j) has properties* (L1) *and* (L2), *write $p := \overline{i_B \circ j}(1)$, and define $v_s := i_\Gamma(s)p$. Then $(i_B \circ j, v)$ is a covariant representation of (A, S, α), and $(i_B \circ j) \times v$ is an isomorphism of $A \times_\alpha S$ onto $p(B \rtimes_\beta \Gamma)p$.*

For the unital case, part (a) is Theorem 2.1 of [12]. Laca proves the existence of (B, Γ, β) using a direct-limit construction, and j is the canonical embedding α^1 of the first copy A_1 of A in the direct limit A_∞. Lemma 4.3 of [13] says that if the endomorphisms are all extendible, then so is $j := \alpha^1$. Laca's proof of uniqueness carries over verbatim. Part (b) is proved for the unital case in [12, Theorem 2.4], and again the proof carries over: the crucial step, which is Lemma 2.3 of [12], is purely representation-theoretic.

In the context of graph algebras, Laca's theorem takes the following form.

Corollary 4.2. *Suppose that S is an Ore semigroup with enveloping group $\Gamma = S^{-1}S$, and β is a free action of Γ on a row-finite k-graph Λ. Suppose that Ω is a saturated subgraph of Λ such that $\beta_u(\Omega) \subset \Omega$ for all $u \in S$ and $\bigcup_{u\in S} \beta_u^{-1}(\Omega) = \Lambda$. Write $\alpha_u := \beta_u|_\Omega$, and set $p := \overline{i_{C^*(\Lambda)}}(p_{\Omega^0})$. Then there is an isomorphism ψ of $C^*(\Omega) \times_{\alpha_*} S$ onto $p(C^*(\Lambda) \rtimes_{\beta_*} \Gamma)p$ such that*

$$\psi(i_{C^*(\Omega)}(s_\omega^\Omega)) = i_{C^*(\Lambda)}(s_\omega^\Lambda) \quad \textit{and} \quad \overline{\psi}(i_S(u)) = i_\Gamma(u)p.$$

Proof. Let $\pi : \Omega \to \Lambda$ be the inclusion. Since Ω is saturated in Λ, Lemma 2.3 implies that π induces an injective extendible homomorphism $\pi_* : C^*(\Omega) \to p_{\Omega^0} C^*(\Lambda) p_{\Omega^0}$ such that $\pi_*(s_\omega^\Omega) = s_\omega^\Lambda$ for $\omega \in \Omega$ and $\overline{\pi_*}(1) = p_{\Omega^0}$. Since each β_u is an automorphism, it is saturated, and we claim that the restriction α_u is saturated as a graph morphism from Ω to Ω. Indeed, if $\omega \in \Omega$ has $r(\omega) \in \alpha_u(\Omega^0)$, say $r(\omega) = \alpha_u(v)$, then

$$r(\beta_u^{-1}(\omega)) = \beta_u^{-1}(r(\omega)) = \beta_u^{-1}(\alpha_u(v)) = \beta_u^{-1}(\beta_u(v)) = v$$

belongs to Ω^0, $\beta_u^{-1}(\omega)$ belongs to Ω because Ω is saturated in Λ, and $\omega = \alpha_u(\beta_u^{-1}(\omega))$ belongs to $\alpha_u(\Omega)$. Now Lemma 2.3 implies that α induces an action α_* of S on $C^*(\Omega)$ by injective extendible endomorphisms.

We will show that the system $(C^*(\Lambda), \Gamma, \beta_*)$ and $j := \pi_*$ have the properties (L1) and (L2) of Theorem 4.1 relative to the semigroup dynamical system $(C^*(\Omega), S, \alpha_*)$. Homomorphisms are determined by what they do on generators, so for $\omega \in \Omega$ and $u \in S$, the calculation

$$\pi_*((\alpha_*)_u(s_\omega^\Omega)) = \pi_*(s_{\alpha_u(\omega)}^\Omega) = s_{\alpha_u(\omega)}^\Lambda = s_{\beta_u(\omega)}^\Lambda = (\beta_*)_u(s_\omega^\Lambda) = (\beta_*)_u(\pi_*(s_\omega^\Omega))$$

implies that $\pi_* \circ (\alpha_*)_u = (\beta_*)_u \circ \pi_*$, which is (L1). Next, note that for $u \in S$ we have

$$(\beta_*)_u^{-1}(\pi_*(C^*(\Omega))) \supset \big\{(\beta_*)_u^{-1}(s_\omega^\Lambda) : \omega \in \Omega\big\} = \big\{s_{\beta_u^{-1}(\omega)}^\Lambda : \omega \in \Omega\big\},$$

which by the hypothesis $\bigcup_{u\in S} \beta_u^{-1}(\Omega) = \Lambda$ implies that $A_0 := \bigcup_{u\in S} (\beta_*)_u^{-1}(\pi_*(C^*(\Omega)))$ contains all the generators of $C^*(\Lambda)$. Thus to check (L2), it is enough to prove that A_0 is a $*$-algebra, and the only non-obvious point is whether A_0 is closed under multiplication. Let $a \in (\beta_*)_u^{-1}(\pi_*(C^*(\Omega)))$ and $b \in (\beta_*)_t^{-1}(\pi_*(C^*(\Omega)))$ for $u, t \in S$. Since S is Ore, there exist $r, w \in S$ such that $ru = wt = x$, say. Since $(\beta_*)_r \circ \pi_* = \pi_* \circ (\alpha_*)_r$, we have $\operatorname{range}(\beta_*)_r \circ \pi_* \subset \operatorname{range} \pi_*$, and

$$(\beta_*)_u^{-1}(\pi_*(C^*(\Omega))) = (\beta_*)_{ru}^{-1} \circ (\beta_*)_r(\pi_*(C^*(\Omega))) \subset (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))).$$

Similarly,

$$(\beta_*)_t^{-1}(\pi_*(C^*(\Omega))) \subset (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))).$$

Since $(\beta_*)_x^{-1}(\pi_*(C^*(\Omega)))$ is an algebra, we have $ab \in (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))) \subset A_0$, as required.

We can now set $v_s := i_\Gamma(s)\overline{i_{C^*(\Lambda)} \circ \pi_*}(1) = i_\Gamma(s)p$, and deduce from Theorem 4.1 that $\psi := (i_{C^*(\Lambda)} \circ \pi_*) \times v$ is an isomorphism of $C^*(\Omega) \times_{\alpha_*} S$ onto $p(C^*(\Lambda) \rtimes_{\beta_*} \Gamma)p$. This isomorphism has the required properties. $\square$

5. Crossed products of the C^*-algebras of skew-product graphs

The action lt of a group Γ on a skew-product $\Lambda \times_\eta \Gamma$ induces an action lt_* of Γ on the graph algebra $C^*(\Lambda \times_\eta \Gamma)$. Kumjian and Pask proved in [9] that the crossed product by this action is stably isomorphic to $C^*(\Lambda)$. Their proof used a groupoid model for the graph algebra and results of Renault about skew-product groupoids, and an explicit isomorphism was constructed in [6]. In the following generalisation of [6, Theorem 3.1], the existence of an isomorphism follows from [10, Theorem 5.7] or [14, Corollary 5.1] (taking $H = G$), but we want an explicit isomorphism.

Theorem 5.1. *Suppose that Λ is a row-finite k graph with no sources, and $\eta : \Lambda \to \Gamma$ is a functor into a group Γ. Then there is an isomorphism ϕ of $C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma$ onto $C^*(\Lambda) \otimes \mathcal{K}(l^2(\Gamma))$ such that*

$$\phi(i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})) = s_\lambda \otimes \chi_g\rho_{\eta(\lambda)} \quad \text{and} \quad \overline{\phi}(i_\Gamma(h)) = 1 \otimes \lambda_h. \tag{5.1}$$

We first show the existence of the homomorphism ϕ. To do this, we verify the following statements in $C^*(\Lambda) \otimes \mathcal{K}(l^2(\Gamma))$:

(1) $S_{(\lambda,g)} := s_\lambda \otimes \chi_g\rho_{\eta(\lambda)}$ is a Cuntz-Krieger $(\Lambda \times_\eta \Gamma)$-family;

(2) if F_n and G_n are increasing sequences of finite subsets of Λ^0 and Γ such that $\Lambda^0 = \bigcup_n F_n$ and $\Gamma = \bigcup_n G_n$, then $\sum_{(v,g)\in F_n\times G_n} S_{(v,g)}$ converges strictly to 1;

(3) $(1 \otimes \lambda_h)S_{(\lambda,g)} = S_{(\lambda,hg)}(1 \otimes \lambda_h)$.

To check (CK1) for the family in (1), we take (v,g) and (w,h) in $(\Lambda \times_\eta \Gamma)^0 = \Lambda^0 \times \Gamma$: then $\eta(v) = \eta(w) = 1$, and $S_{(v,g)}S_{(w,h)} = s_v s_w \otimes \chi_g\chi_h$, which gives (CK1). Next, suppose that (λ,g) and (μ,h) are composable, so that $s(\lambda) = r(\mu)$ and $g\eta(\lambda) = h$. Then $\rho_k\chi_{gk} = \chi_g\rho_k$ implies that

$$\begin{aligned} S_{(\lambda,g)}S_{(\mu,h)} &= (s_\lambda \otimes \chi_g\rho_{\eta(\lambda)})(s_\mu \otimes \chi_h\rho_{\eta(\mu)}) = (s_\lambda s_\mu) \otimes (\chi_g\rho_{\eta(\lambda)}\chi_{g\eta(\lambda)}\rho_{\eta(\mu)}) \\ &= s_{\lambda\mu} \otimes (\chi_g\chi_g\rho_{\eta(\lambda)}\rho_{\eta(\mu)}) = s_{\lambda\mu} \otimes (\chi_g\rho_{\eta(\lambda\mu)}) = S_{(\lambda\mu,g)}, \end{aligned}$$

which is (CK2). A similar calculation gives (CK3), and a calculation using the Cuntz-Krieger relation for $\{s_\lambda\}$ gives (CK4). We have now proved item (1).

Next observe that

$$\sum_{(v,g)\in F_n\times G_n} S_{(v,g)} = \Big(\sum_{v\in F_n} s_v\Big) \otimes \Big(\sum_{g\in G_n} \chi_g\Big) = a_n \otimes b_n,$$

say, and then (2) holds because $\{a_n\}$ and $\{b_n\}$ are approximate identities for $C^*(\Lambda)$ and $\mathcal{K}(l^2(\Gamma))$. Finally, a calculation using $\lambda_h\chi_g = \chi_{hg}\lambda_h$ and $\rho_g\lambda_h = \lambda_h\rho_g$ gives (3).

Item (1) implies that there is a homomorphism π_S from $C^*(\Lambda\times_\eta\Gamma)$ to $C^*(\Lambda)\otimes\mathcal{K}(l^2(\Gamma))$ taking $s_{(\lambda,g)}$ to $S_{(\lambda,g)}$, and (2) then says that π_S is nondegenerate. Item (3) implies that $(\pi_S, 1 \otimes \lambda)$ is a covariant representation of $(C^*(\Lambda \times_\eta \Gamma), \Gamma, \mathrm{lt}_*)$ in $C^*(\Lambda) \otimes \mathcal{K}(l^2(\Gamma))$, and $\phi := \pi_S \times (1 \otimes \lambda)$ satisfies (5.1). The image of each spanning element $s_{(\lambda,g)}s^*_{(\mu,k)}i_\Gamma(h)$ belongs to $C^*(\Lambda) \otimes \mathcal{K}(l^2(\Gamma))$, and hence ϕ has range in $C^*(\Lambda) \otimes \mathcal{K}(l^2(\Gamma))$.

To see that ϕ is surjective, we note that the range of ϕ contains every element $s_\lambda \otimes \chi_g\rho_{\eta(\lambda)}\lambda_h$. The operator $\chi_g\rho_{\eta(\lambda)}\lambda_h$ is the rank-one operator $e_g \otimes \overline{e}_{h^{-1}g\eta(\lambda)}$, and for each λ, each matrix unit $e_p \otimes \overline{e}_q$ arises for a suitable choice of g and h. Thus the range of ϕ contains every $s_\lambda \otimes (e_p \otimes \overline{e}_q)$, and every

$$s_\lambda s_\mu^* \otimes (e_p \otimes \overline{e}_q) = \big(s_\lambda \otimes (e_p \otimes \overline{e}_q)\big)\big(s_\mu \otimes (e_q \otimes \overline{e}_q)\big)^*;$$

since these elements span a dense $*$-subalgebra of $C^*(\Lambda)\otimes\mathcal{K}(l^2(\Gamma))$, and homomorphisms of C^*-algebras have closed range, we deduce that ϕ is surjective.

To prove that ϕ is injective, we will construct a left inverse for ϕ.

Lemma 5.2. *Suppose that $\{y_g : g \in \Gamma\}$ is a set of mutually orthogonal projections in a C^*-algebra D, and $u : \Gamma \to UM(D)$ is a homomorphism such that*

$$u_h y_g = y_{hg} u_h. \tag{5.2}$$

Then there is a homomorphism $y \times u : \mathcal{K}(l^2(\Gamma)) \to D$ such that $y \times u(\lambda_h \chi_g) = u_h y_g$.

Proof. Observe that $e_{g,h} := u_g y_1 u_h^*$ is a set of matrix units in D, and thus Corollary A.9 of [15] gives a homomorphism $y\times u : \mathcal{K}(l^2(\Gamma)) \to D$ such that $(y\times u)(e_g\otimes\overline{e}_h) = u_g y_1 u_h^*$. Now verify that $\lambda_h\chi_g = e_{hg}\otimes\overline{e}_g$, and we have $y \times u(\lambda_h\chi_g) = u_{hg}y_1u_g^* = u_hy_g$. □

Lemma 5.3. *Suppose that Λ, Γ and η are as in Theorem 5.1.*

(a) *The elements*

$$y_g := \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(p_{\Lambda^0\times\{g\}}) \quad \text{and} \quad u_h := i_\Gamma(h)$$

of $M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$ satisfy (5.2). The homomorphism $y \times u$ from Lemma 5.2 is nondegenerate; the elements $w_k := \overline{y \times u}(\rho_k)$ commute with u_h and satisfy

$$w_k y_g = y_{gk^{-1}} w_k. \tag{5.3}$$

(b) *The partial isometries*

$$T_\lambda := \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})w_{\eta(\lambda)}^{-1}$$

commute with every y_g, u_h and w_k.

(c) *$\{T_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in $M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$.*

Proof. We choose increasing sequences of finite subsets G_n of Λ^0 and H_n of Γ such that $\Lambda^0 = \bigcup_n G_n$ and $\Gamma = \bigcup_n H_n$. Then the strict continuity of $\overline{i_{C^*(\Lambda\times_\eta\Gamma)}}$ implies that

$$i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,g)}\Big) \to y_g \ \text{ strictly.}$$

For each n, the covariance of $(i_{C^*(\Lambda\times_\eta\Gamma)}, i_\Gamma)$ implies that

$$u_h i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,g)}\Big) = i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,hg)}\Big)u_h,$$

and since the right-hand side converges strictly to $y_{hg}u_h$, (5.2) follows.

Since $r(\alpha, g) = (r(\alpha), g)$ belongs to $\Lambda^0\times\{g\}$, the formula (2.1) shows that $s_{(\alpha,g)}s^*_{(\beta,h)} = y_g s_{(\alpha,g)}s^*_{(\beta,h)}$, and this implies that $y \times u$ is nondegenerate. So the formula for w_k makes sense. It has the described properties because ρ_k commutes with λ_h and satisfies $\rho_k\chi_g = \chi_{gk^{-1}}\rho_k$. We have now proved (a).

The last assertion in Lemma 2.1 implies that

$$y_g T_\lambda = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(p_{\Lambda^0\times\{g\}}s_{\{\lambda\}\times\Gamma})w_{\eta(\lambda)}^{-1} = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})w_{\eta(\lambda)}^{-1}. \tag{5.4}$$

On the other hand, (5.3) implies that $w_{\eta(\lambda)}^{-1}y_g = y_{g\eta(\lambda)}w_{\eta(\lambda)}^{-1}$, and thus

$$T_\lambda y_g = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma}p_{\Lambda^0\times\{g\eta(\lambda)\}})w_{\eta(\lambda)}^{-1},$$

which since $s(\lambda, g) = (s(\lambda), g\eta(\lambda))$ is the same as the right-hand side of (5.4). Thus y_g commutes with T_λ.

To see that u_h commutes with T_λ, we realise $s_{\{\lambda\}\times\Gamma}$ as the strict limit of the finite sums $s_{\{\lambda\}\times H_n} := \sum_{g\in H_n} s_{(\lambda,g)}$. Then T_λ is the strict limit of $t_n := i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times H_n})$, and u_hT_λ is the strict limit of u_ht_n. Covariance implies that

$$u_ht_n = i_\Gamma(h)i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times H_n}) = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times hH_n})u_h, \tag{5.5}$$

and since the limit $s_{\{\lambda\}\times\Gamma}$ is independent of the choice of increasing subsets, the right-hand side of (5.5) converges strictly to $T_\lambda u_h$. Thus $u_hT_\lambda = T_\lambda u_h$. Since T_λ commutes with everything in the ranges of $y\times u$ and $\overline{y\times u}$, including w_k, we have proved (b).

Since $\eta(v) = 1$ for every vertex v, the relation (CK1) for $\{T_\lambda\}$ follows from the assertion $s_Vs_W = s_{VW}$ in Lemma 2.1. For (CK2), we suppose λ and μ are composable in Λ. Then because $w_{\eta(\lambda)}^{-1}$ and T_μ commute, we have

$$T_\lambda T_\mu = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})T_\mu w_{\eta(\lambda)}^{-1} = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma}s_{\{\mu\}\times\Gamma})(w_{\eta(\lambda)\eta(\mu)})^{-1},$$

and the right-hand side reduces to $T_{\lambda\mu}$ because $s_Vs_W = s_{VW}$, $k\mapsto w_k$ is a homomorphism, and η is a functor. For (CK3), we need to compute

$$T_\lambda^*T_\lambda = w_{\eta(\lambda)}\overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma}^*s_{\{\lambda\}\times\Gamma})w_{\eta(\lambda)}^{-1}.$$

From (2.1) and the adjoint of (2.2), we deduce that $(s_{\{\lambda\}\times\Gamma}^*s_{\{\lambda\}\times\Gamma})(s_{(\alpha,g)}s_{(\beta,h)}^*)$ vanishes unless $r(\alpha) = s(\lambda)$, and then is $s_{(\alpha,g)}s_{(\beta,h)}^*$; thus left multiplication by $s_{\{\lambda\}\times\Gamma}^*s_{\{\lambda\}\times\Gamma}$ is the same as left multiplication by $s_{\{s(\lambda)\}\times\Gamma}$, and $s_{\{\lambda\}\times\Gamma}^*s_{\{\lambda\}\times\Gamma} = s_{\{s(\lambda)\}\times\Gamma}$. Thus $T_\lambda^*T_\lambda = w_{\eta(\lambda)}T_{s(\lambda)}w_{\eta(\lambda)}^{-1}$, and since $w_{\eta(\lambda)}$ commutes with $T_{s(\lambda)}$, we recover (CK3). For (CK4) we fix $v\in\Lambda^0$ and $n\in\mathbb{N}^k$, and then

$$\sum_{\lambda\in v\Lambda^n} T_\lambda T_\lambda^* = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}\Big(\sum_{\lambda\in v\Lambda^n} s_{\{\lambda\}\times\Lambda}s_{\{\lambda\}\times\Lambda}^*\Big);$$

a calculation using the formulas in Lemma 2.1 shows that left multiplication by the inside sum is the same as left multiplication by $s_{\{v\}\times\Gamma}$, and this gives (CK4). □

Proof of Theorem 5.1. In the paragraphs following the statement, we constructed ϕ and showed it is surjective. For injectivity, we consider the homomorphism $y\times u : \mathcal{K}(l^2(\Gamma)) \to M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ associated to the elements y_g and u_h described in Lemma 5.3(a), and the homomorphism π_T of $C^*(\Lambda)$ into $M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ given by the Cuntz-Krieger family $\{T_\lambda\}$ of Lemma 5.3. Lemma 5.3(b) implies that π_T and $y\times u$ have commuting ranges, and hence give a homomorphism $\theta := \pi_T\otimes(y\times u)$ of $C^*(\Lambda)\otimes\mathcal{K}(l^2(\Gamma))$ into $M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ such that $\theta(a\otimes k) = \pi_T(a)(y\times u)(k)$ (by [17, Theorem B.27]).

Finally we compute, using in particular the formula (5.3):

$$\begin{aligned}\theta\circ\phi\big(i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h)\big) &= \theta\big(s_\lambda\otimes\chi_g\rho_{\eta(\lambda)}\lambda_h\big)\\ &= \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})w_{\eta(\lambda)}^{-1}y_gw_{\eta(\lambda)}u_h = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})y_{g\eta(\lambda)}w_{\eta(\lambda)}^{-1}w_{\eta(\lambda)}u_h\\ &= \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}\big(s_{\{\lambda\}\times\Gamma}p_{\Lambda^0\times\{g\eta(\lambda)\}}\big)i_\Gamma(h) = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h).\end{aligned}$$

Since the elements $i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h)$ generate the crossed product, this proves that $\theta\circ\phi$ is the identity, and in particular that ϕ is injective. □

6. The main theorem

Theorem 6.1. *Suppose that Σ is a row-finite k-graph with no sources, and α is a free action of an Ore semigroup S on Σ which admits a fundamental domain F. Let $q : \Sigma \to S\backslash\Sigma$ be the quotient map, and define $c : S\backslash\Sigma \to F$, $\eta : S\backslash\Sigma \to S$, $\xi : \Sigma \to S$ by*

$$q(c(\lambda)) = \lambda, \quad s(c(\lambda)) = \alpha_{\eta(\lambda)}(c(s(\lambda))) \quad \text{and} \quad \sigma = \alpha_{\xi(\sigma)}(c(q(\sigma))). \tag{6.1}$$

Then there is an isomorphism ψ of $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(l^2(S))$ such that

$$\psi(i_{C^*(\Sigma)}(s_\sigma^\Sigma)) = s_{q(\sigma)} \otimes (\chi_{\xi(\sigma)}\rho_{\eta(q(\sigma))}|_{l^2(S)}) \quad \text{and} \quad \overline{\psi}(i_S(u)) = 1 \otimes \lambda_u^S.$$

We need a general lemma about tensor products of multipliers.

Lemma 6.2. *Suppose that A and B are C^*-algebras. For each $m \in M(A)$ and $n \in M(B)$ there is a multiplier $m \otimes_{\max} n$ of $A \otimes_{\max} B$ such that*

$$(m \otimes_{\max} n)(a \otimes b) = ma \otimes nb \quad \text{and} \quad (a \otimes b)(m \otimes_{\max} n) = am \otimes bn. \tag{6.2}$$

The map $\iota : (m, n) \mapsto m \otimes_{\max} n$ is strictly continuous in the following weak sense: if $m_i \to m$ strictly in $M(A)$, $n_i \to n$ strictly in $M(B)$, and both $\{m_i\}$ and $\{n_i\}$ are bounded, then $m_i \otimes_{\max} n_i \to m \otimes_{\max} n$ strictly.

Proof. Consider the canonical maps $j_A : A \to M(A \otimes_{\max} B)$ and $j_B : B \to M(A \otimes_{\max} B)$, as in, for example, [17, Theorem B.27]. Then j_A and j_B are nondegenerate homomorphisms with commuting ranges such that $j_A(a)j_B(b) = a \otimes b$ [17, Theorem B.27(a)]. The extensions $\overline{j_A}$ to $M(A)$ and $\overline{j_B}$ to $M(B)$ also have commuting ranges, and hence there is a homomorphism $\overline{j_A} \otimes_{\max} \overline{j_B}$ of $M(A) \otimes_{\max} M(B)$ into $M(A \otimes_{\max} B)$ such that $\overline{j_A} \otimes_{\max} \overline{j_B}(m \otimes n) = \overline{j_A}(m)\overline{j_B}(n)$. We define $m \otimes_{\max} n := \overline{j_A} \otimes_{\max} \overline{j_B}(m \otimes n)$ Then

$$\begin{aligned}(m \otimes_{\max} n)(a \otimes b) &= (\overline{j_A}(m)\overline{j_B}(n))(i_A(a)i_B(b)) = (\overline{j_A}(m)j_A(a))(\overline{j_B}(n)j_B(b)) \\ &= j_A(ma)j_B(nb) = ma \otimes nb,\end{aligned}$$

and similarly on the other side. Since $\overline{j_A}$ and $\overline{j_B}$ are strictly continuous, $\overline{j_A}(m_i) \to \overline{j_A}(m)$ and $\overline{j_B}(n_i) \to \overline{j_B}(n)$, and the strict continuity of multiplication on bounded sets implies that $m_i \otimes_{\max} n_i = \overline{j_A}(m_i)\overline{j_B}(n_i)$ converges to $\overline{j_A}(m)\overline{j_B}(n) = m \otimes_{\max} n$. □

Remark 6.3. When we apply Lemma 6.2, at least one of A or B is nuclear, and $A \otimes_{\max} B$ coincides with the usual spatial tensor product; then, since there is at most one multiplier satisfying (6.2), $m \otimes_{\max} n$ coincides with the usual spatially defined $m \otimes n$. However, $M(A)$ and $M(B)$ need not be nuclear (even for $B = \mathcal{K}(\mathcal{H})$!), so this observation merely says that $\overline{j_A} \otimes_{\max} \overline{j_B}$ on $M(A) \otimes_{\max} M(B)$ factors through the spatial tensor product.

Proof of Theorem 6.1. Our Gross-Tucker theorem (Theorem 3.2) describes an isomorphism ϕ of Σ onto the skew product $(S\backslash\Sigma) \times_\eta S$ such that $\phi \circ \alpha_t = \mathrm{lt}_t \circ \phi$. The induced isomorphism ϕ_* of $C^*(\Sigma)$ onto $C^*((S\backslash\Sigma) \times_\eta S)$ satisfies $\phi_* \circ \alpha_* = \mathrm{lt}_* \circ \phi_*$, and hence induces an isomorphism ψ_1 of $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*((S\backslash\Sigma) \times_\eta S) \times_{\mathrm{lt}_*} S$ satisfying

$$\psi_1(i_{C^*(\Sigma)}(s_\sigma^\Sigma)) = i_{C^*((S\backslash\Sigma)\times_\eta S)}(s_{(q(\sigma),\xi(\sigma))}) \quad \text{and} \quad \overline{\psi_1}(i_S(u)) = i_S(u).$$

We want to apply Corollary 4.2 with $\Lambda = (S\backslash\Sigma) \times_\eta \Gamma$, $\Omega = (S\backslash\Sigma) \times_\eta S$ and $\beta = \mathrm{lt}$. The subgraph Ω is saturated, because $r(\lambda, g) = (r(\lambda), g)$ belongs to Ω^0 precisely when $g \in S$, in which case (λ, g) belongs to Ω. We trivially have $\mathrm{lt}_t(\Omega) \subset \Omega$ for $t \in S$, and because $\Gamma = S^{-1}S$, every $g \in \Gamma$ can be written as $t^{-1}u$ for $t, u \in S$, and then every

$(\lambda, g) = \mathrm{lt}_t^{-1}(\lambda, u)$ belongs to $\bigcup_{t\in S} \mathrm{lt}_t^{-1}(\Omega)$. The restriction of lt_u to Ω is just the lt_u in the previous paragraph. So with $p := \overline{i_{C^*((S\backslash\Sigma)\times_\eta S)}}(p_{(S\backslash\Sigma)^0\times S})$, Corollary 4.2 gives an isomorphism ψ_2 of $C^*((S\backslash\Sigma)\times_\eta S)\times_{\mathrm{lt}_*} S$ onto $p\big(C^*((S\backslash\Sigma)\times_\eta \Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ such that

$$\psi_2\big(i_{C^*((S\backslash\Sigma)\times_\eta S)}(s_{(\lambda,t)})\big) = i_{C^*((S\backslash\Sigma)\times_\eta \Gamma)}(s_{(\lambda,t)}) \text{ and } \overline{\psi_2}(i_S(u)) = i_\Gamma(u)p. \tag{6.3}$$

Theorem 5.1 gives an isomorphism ϕ of $C^*((S\backslash\Sigma)\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma$ onto $C^*(S\backslash\Sigma)\otimes\mathcal{K}(l^2(\Gamma))$ such that

$$\phi\big(i_{C^*((S\backslash\Sigma)\times_\eta\Gamma)}(s_{(\lambda,g)})\big) = s_\lambda\otimes\chi_g\rho_{\eta(\lambda)} \text{ and } \overline{\phi}(i_\Gamma(h)) = 1\otimes\lambda_h.$$

Since ϕ is an isomorphism, it extends to the multiplier algebra, and restricts an isomorphism of $p\big(C^*((S\backslash\Sigma)\times_\eta \Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ onto $\overline{\phi}(p)\big(C^*((S\backslash\Sigma)\times_\eta \Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)\overline{\phi}(p)$.

Again write $(S\backslash\Sigma)^0$ and S as increasing unions $\bigcup_n G_n$ and $S = \bigcup_n H_n$ of finite subsets. Then $p_{(S\backslash\Sigma)^0\times S}$ is by definition the strict limit of $p_n := \sum_{(w,u)\in G_n\times H_n} p_{(w,u)}$ (see Lemma 2.1). Thus, since $\eta(w) = 1$ for every vertex w, $\overline{\phi}(p)$ is the strict limit of

$$\begin{aligned}\phi\big(i_{C^*((S\backslash\Sigma)\times_\eta\Gamma)}(p_n)\big) &= \sum_{(w,u)\in G_n\times H_n} \phi\big(i_{C^*((S\backslash\Sigma)\times_\eta\Gamma)}(p_{(w,u)})\big)\\ &= \sum_{(w,u)\in G_n\times H_n} p_w\otimes\chi_u\\ &= \Big(\sum_{w\in G_n} p_w\Big)\otimes\Big(\sum_{u\in H_n}\chi_u\Big).\end{aligned}$$

Since $\sum_{w\in G_n} p_w$ and $\sum_{u\in H_n}\chi_u$ converge strictly to $1_{M(C^*(S\backslash\Sigma))}$ and χ_S, the assertion about strict continuity in Lemma 6.2 implies that $\overline{\phi}(p) = 1_{M(C^*(S\backslash\Sigma))}\otimes\chi_S$. A calculation on elementary tensors shows that

$$(1\otimes\chi_S)(C^*(S\backslash\Sigma)\otimes\mathcal{K}(l^2(\Gamma))(1\otimes\chi_S) = C^*(S\backslash\Sigma)\otimes\chi_S\mathcal{K}(l^2(\Gamma))\chi_S.$$

Since we are identifying $l^2(S)$ with a subspace of $l^2(\Gamma)$ and χ_S is then the orthogonal projection of $l^2(\Gamma)$ onto $l^2(S)$, $\chi_S\mathcal{K}(l^2(\Gamma)\chi_S$ is naturally identified with $\mathcal{K}(l^2(S))$. When we make this identification, $\chi_S\lambda_u\chi_S$ is the generator $\lambda_u^S := \lambda_u\chi_S$ of the Toeplitz representation of S on $l^2(S)$. Thus restricting ϕ gives an isomorphism

$$\psi_3 : p\big(C^*((S\backslash\Sigma)\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p \to C^*(S\backslash\Sigma)\otimes\mathcal{K}(l^2(S))$$

such that for t and u in S,

$$\psi_3\big(p\, i_{C^*((S\backslash\Sigma)\times_\eta\Gamma)}(s_{(\lambda,t)})p\big) = s_\lambda\otimes(\chi_t\rho_{\eta(\lambda)})|_{l^2(S)} \text{ and } \overline{\psi_3}(i_\Gamma(u)) = \lambda_u^S$$

(notice that although $\rho_{\eta(\lambda)}$ does not leave $l^2(S)$ invariant, the product $\chi_t\rho_{\eta(\lambda)}$ does).

Now $\psi := \psi_3\circ\psi_2\circ\psi_1$ has the required properties. □

Corollary 6.4. *Suppose that Σ is a 2-graph, α is a free action of an Ore semigroup S on Σ which admits a fundamental domain F. Then $C^*(\Sigma)\times_{\alpha_*} S$ is purely infinite and simple if and only if $C^*(S\backslash\Sigma)$ is purely infinite and simple.*

Proof. Both simplicity and pure-infiniteness are preserved by stable isomorphism (by [19, Proposition 4.1.8] for pure infiniteness), so the result follows from Theorem 6.1. □

The point of the Corollary is that $S\backslash\Sigma$ is smaller than Σ, hence is likely to be more tractable, and we have criteria for deciding whether $C^*(S\backslash\Sigma)$ is purely infinite and simple. We illustrate with a example which is similar to one studied in [7].

Example 6.5. We consider the graph $\mathbb{F}_\theta^2$ of [8, 3] associated to the permutation θ of $\{1,2,3\}\times\{1,2,3\}$ defined by

$$\theta(2,j)=(1,j),\ \theta(1,j)=(2,j),\ \theta(3,j)=(3,j) \text{ for } j=1,3, \text{ and}$$
$$\theta(i,2)=(i,2) \ \text{ for } i=1,2,3.$$

As in [7, Example 5.7], there is a functor $c:\mathbb{F}_\theta^2\to\mathbb{Z}^2$ such that $c(g_3)=(0,1)$, $c(f_3)=(1,0)$, and $c(f_i)=(0,0)$, $c(g_i)=(0,0)$ for $i=1,2$. Since this functor takes values in $\mathbb{N}^2$, we can apply Corollary 6.4 to the action lt of $\mathbb{N}^2$ on $\mathbb{F}_\theta^2\times_c\mathbb{N}^2$, for which the quotient graph is $\mathbb{F}_\theta^2$. It is shown in [7, Example 5.7] that $\mathbb{F}_\theta^2$ is aperiodic, and since $\mathbb{F}_\theta^2$ has a single vertex, it is trivially cofinal. Thus $C^*(\mathbb{F}_\theta^2)$ is simple by [18, Theorem 3.4], and purely infinite by [20, Proposition 8.8]. Thus Corollary 6.4 implies that $C^*(\mathbb{F}_\theta^2\times_c\mathbb{N}^2)\times_{\mathrm{lt}_*}\mathbb{N}^2$ is purely infinite and simple. On the other hand, the discussion in [7, Example 3.5] shows that $C^*(\mathbb{F}_\theta^2\times_c\mathbb{N}^2)$ has many ideals.

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